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SPECIFIC FEATURES OF THE NONLINEARLY ELASTIC BEHAVIOR OF CYLINDRICAL COMPRESSIBLE BODIES IN TORSION

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The torsion problem of a cylinder with a circular transverse cross section twisted by end moments that are equal in magnitude and opposite in direction is considered for various models of nonlinearly elastic compressible media. The problem is solved by the semi-inverse method of elasticity theory. The Poynting effect, which consists of variation in the length of a shaft in torsion, is treated qualitatively and quantitatively. The results of the numerical and asymptotic (only terms that are quadratic relative to the displacement gradient are conserved) solutions for various models of the nonlinearly elastic behavior of materials are compared. An analysis of the results shows that in some cases, the quasilinear model is not applicable for studying the behavior of nonlinearly elastic compressible media.

The phenomenon of variation of the length of an elastic cylinder in torsion was discovered experimentally and described by Poynting in the early twentieth century. Quantitatively, this effect is manifested weakly: when the torsion angles are approximately 15–20° per unit length, the relative elongation of the sample does not exceed 0.01. However, in manufacturing precision measuring devices and in determining experimentally the elastic constants of materials, the influence of the Poynting effect should be taken into account.

The phenomenon discovered by Poynting can be explained by means of the nonlinear torsion problem.

The torsion problem of a compressible (changing its volume upon deformation) cylinder with allowance for axial elongation was treated both in known books dealing with continuum mechanics [1, 2] and in recent studies. In particular, M. Chen and Z. Chen [3] analyzed this problem with the use of asymptotic methods, and Koczyk and Weese [4] solved it by the finite-element method; the torsion problem of circular cylinders was analyzed numerically by Zubov [5] and Gavrilyachenko [6]. However, most of these studies considered concrete models of an elastic material.

The goal of the present study is to estimate quantitatively and qualitatively the Poynting effect with the use of various governing relations for isotropic compressible materials and compare the behavior of their models.

Governing Relations. We introduce the reference (unstrained) and real (strained) configurations of the medium. The radius vector of a material point in the real configuration is denoted by $\boldsymbol{R}$. The second-rank tensor C, which is called a strain gradient, is specified by the relation $C = \operatorname{grad} \boldsymbol{R}$, where grad is the gradient operator in the basis of the reference configuration.

The Piola stress tensor defined in the reference configuration is expressed in terms of the "true" stress tensor T as follows [1]:

$$D = (C^{\mathrm{t}})^{-1} T \det C.$$

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Here the superscript "t" denotes transposition.

The model of an elastic material is characterized by the form of the function of specific potential strain energy W. For an isotropic material, the latter can be defined [1] as a function of the basic invariants I_k of the Cauchy–Green strain measure $G = CC^{\mathrm{t}}$.

The governing relations for an isotropic material that possesses the potential W have the form

$$D = \frac{\partial W}{\partial C} = 2\frac{\partial W}{\partial G}C \qquad [W = W(I_1, I_2, I_3).] \tag{1}$$

Formulation of the Problem and the Method of Its Solutions. Let an elastic cylinder (shaft) with a circular transverse cross section be twisted by the end moments M_0 that are equal in magnitude and opposite in direction. Its length l before deformation is assumed to be quite large, and the side surface is assumed to be free from loading. The external radius and the radius of a hollow are denoted by r_1 and r_0, respectively.

We assume that after deformation the cylinder becomes a cylinder and, hence, the reference and real configurations can be conveniently considered in a cylindrical coordinate system. The cylindrical coordinates of a material particle in the reference and real configurations are denoted by r, φ, and z and R, Φ, and Z, respectively, and the appropriate orthonormalized-basis vector by $\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$, and $\boldsymbol{e}_z$ and $\boldsymbol{e}_R$, $\boldsymbol{e}_\Phi$, and $\boldsymbol{e}_Z$, respectively.

The torsion of a cylindrical shaft is described by the following transformation of the reference configuration to a real one:

$$R = P(r), \quad \Phi = \varphi + \psi z, \quad Z = \gamma z, \quad r_0 \leqslant r \leqslant r_1, \quad 0 \leqslant z \leqslant l. \tag{2}$$

Here $P(r)$ is a desired function that should be determined, ψ is the torsion angle referred to unit length, and γ is the elongation of the shaft during torsion. The strain gradient, the Cauchy–Green strain measure, and its basic invariants are determined from this transformation. After that, the Piola stress tensor is expressed from the equation of state (1).

We assume that the stress state is created by surface forces, whereas the effect of mass forces is negligible. Then, the equilibrium equation expressed in terms of the Piola stress tensor takes the form

$$\operatorname{div} D = 0. \tag{3}$$

Here div is the divergence operator in the coordinates of the reference configuration.

In the reference configuration, in the absence of loading on a part of the body surface with the normal $\boldsymbol{n}$, the relation $\boldsymbol{n} \cdot D = 0$ holds. With allowance for this, the boundary condition on the side surface with the unit normal $\boldsymbol{e}_r$ is written in the form

$$\boldsymbol{e}_r \cdot D = 0. \tag{4}$$

The expressions for the axial force Q and the torsional moment M that act in the transverse cross section of the shaft being twisted have the form

$$Q = \int\limits_{S^*} \sigma_Z \, dS^* = \int\limits_{S} D_{zZ} \, dS, \qquad M_k = \int\limits_{S^*} R\tau_{\Phi Z} \, dS^* = \int\limits_{S} RD_{\varphi Z} \, dS, \tag{5}$$

where S^* and S are the cross-sectional areas in the real and reference configurations, respectively, $\sigma_Z = \boldsymbol{e}_Z \cdot T \cdot \boldsymbol{e}_Z$ and $\tau_{\Phi Z} = \boldsymbol{e}_\Phi \cdot T \cdot \boldsymbol{e}_Z$ are the components of the Cauchy stress tensor, and $D_{\varphi Z} = \boldsymbol{e}_\varphi \cdot D \cdot \boldsymbol{e}_Z$ and $D_{zZ} = \boldsymbol{e}_z \cdot D \cdot \boldsymbol{e}_Z$ are the components of the Piola stress tensor.

Since the but-ends of the shaft considered are not fixed, an axial force does not appear during torsion; therefore, with Eq. (5) taken into account, the boundary conditions at the but-ends are written in the form

$$\int\limits_{S^*} \sigma_Z \, dS^* = \int\limits_{S} D_{zZ} \, dS = 0, \qquad \int\limits_{S^*} R\tau_{\Phi Z} \, dS^* = \int\limits_{S} RD_{\varphi Z} \, dS = M_0. \tag{6}$$

The second-order nonlinear boundary-value equilibrium problem (3), (4) relative to the function $P(r)$ is solved numerically or analytically.

After the function $P(r)$ is determined, an expression for the axial force that depends on two parameters in this case, namely, the relative elongation of the shaft γ and the torsion angle per unit length ψ, is constructed.

To assess the Poyting effect, it suffices to choose the values of γ at which the axial force vanishes for given values of ψ by means of the first condition in (6). This scheme of constructing the solution is called the semi-inverse [1] method of elasticity theory. This method is quite efficient but is applicable to a narrow class of problems that is limited to a set of simple geometrical transformations of conic bodies.

The semi-inverse method is treated in the literature in detail. We expound this method in detail here to show the high degree of its algorithmization. Derivation of the boundary-value problem for a real energy function W is a chain of cumbersome transformations. However, algorithmization makes the use of the means of computer algebra effective in similar problems. For particular models of nonlinearly elastic media, the boundary-value problems given below were generated automatically and analyzed numerically by a program developed by the authors in Maple V medium for Windows [6].

Below, we consider a number of models of nonlinearly elastic compressible media.

Two-Constant, Physically Linear Model. For this model, the function of specific potential strain energy is given by the expression

$$W = \Big(\frac{\lambda}{2} + \mu\Big) j_1^2 - 2\mu j_2 = \frac{\lambda + 2\mu}{8}\, I_1^2 - \frac{3\lambda + 2\mu}{4}\, I_1 + \frac{3}{8}\,(3\lambda + 2\mu),$$

where j_k are the basic invariants of the Cauchy–Green strain tensor $K = (1/2)(G - E)$ (E is a unit tensor) and λ and μ are the Lamé constants. Hereafter, $I_k = I_k(G)$. The boundary-value equilibrium problem has the form

$$P'' = \frac{1}{\eta}\Big(P^3\Big(\frac{(\psi^2 r^2 + 1)^2}{r^2} + \nu\psi^2(r^2\psi^2 - 2) - \frac{\nu}{r^2}\Big) - P'^3 r(1 - \nu) - P'^2 P\nu(\psi^2 r^2 - 1)$$

$$- P^2 P'\nu\Big(r\psi^2 - \frac{1}{r}\Big) - P' r(\nu\gamma^2 - \nu - 1) - P\Big(\psi^2\gamma^2(\nu - 1) + \psi^2 r^2(\nu + 1) + \nu(1 - \gamma^2) + \frac{1}{r}\Big)\Big),$$

$$\eta = P^2\nu(r^2\psi^2 - 1) + 3P'^2 r^2(1 - \nu) + r^2(\nu\gamma^2 - \nu - 1),$$

$$P' r^2(\nu\gamma^2 - \nu - 1) + P'^3 r^2(1 - \nu) + P' P^2\nu(1 + r^2\psi^2) = 0, \qquad r = r_0, r_1,$$

where $\nu = \lambda/(2(\lambda + \mu))$.

Murnaghan's Five-Constant Model [1]. For this model, the function of specific potential strain energy is specified in the form

$$W = \frac{1}{4}\Big(- 3\lambda - 2\mu + \frac{9}{2}\, l + \frac{n}{2}\Big) I_1 + \frac{1}{8}\,(\lambda + 2\mu - 3l - 2m) I_1^2 + \frac{1}{4}\Big(- 2\mu + 3m - \frac{n}{2}\Big) I_2$$

$$- \frac{m}{4}\, I_1 I_2 + \frac{1}{24}\,(l + 2m) I_1^3 + \frac{n}{2}\,(I_3 - 1) + \frac{3}{8}\,(3\lambda + 2\mu) - \frac{9}{8}\, l,$$

where l, m, and n are the Murnaghan constants. For $l = m = n = 0$, this model becomes a model of a physically linear material. The values of the Lamé and Murnaghan constants are given in [1] for many materials.

The boundary-value equilibrium problem for this model is omitted here because of its cumbersome form.

Blatz and Ko's Three-Constant Model [1]. For this model, the function of specific potential strain energy is given by the expression

$$W = \frac{\mu}{2}\,(1 - \beta)\Big(\frac{I_2}{I_3} + \frac{1}{\alpha}\,(I_3^{\alpha} - 1) - 3\Big) + \frac{\mu}{2}\,\beta\Big(I_1 + \frac{1}{\alpha}\,(I_3^{-\alpha} - 1) - 3\Big).$$

Here α, β, and μ are constants of the materials. Upon small deformations, this model becomes a model of a linearly elastic material with Lamé constants $\lambda = 2\mu\alpha$ and μ.

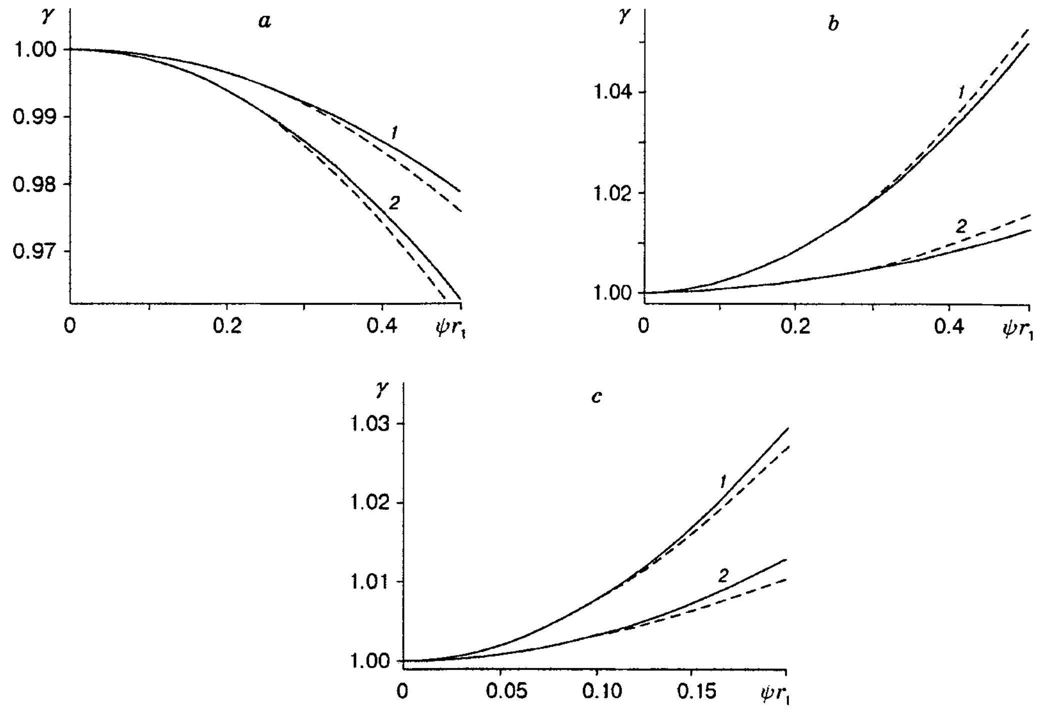

Fig. 1

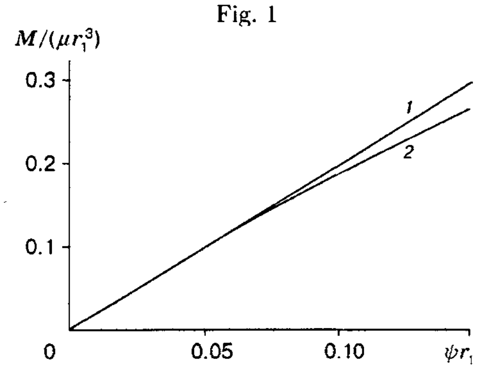

Fig. 2

The following simplified variant of the model is obtained for $\alpha = 1/2$ and $\beta = 0$:

$$W = \frac{1}{2}\mu\Big(\frac{I_2}{I_3} + 2\sqrt{I_3} - 5\Big). \tag{7}$$

For the potential (7), the boundary-value equilibrium problem can be written in the form

$$P'' = \frac{1}{3}\Big(\frac{P'}{r} - \frac{r^2 P'^4}{P^3}\Big), \qquad P = \frac{r}{P'^3\gamma}, \qquad r = r_0, r_1.$$

where $W = W(j_1, j_2, j_3)$, $W_{,k} = \partial W/\partial j_k$, and $W_{,ks} = \partial^2 W/\partial j_k\,\partial j_s$.

It has the analytical solution

$$P(r) = \gamma^{-1/4} r, \qquad \gamma = \Big(1 + \frac{J_p}{S}\psi^2\Big)^{2/5}.$$

Hereafter, S is the cross-sectional area of the shaft before deformation and J_p is the polar moment of this cross section.

For a hypothetical ($\alpha = 1/2$ and $\beta = 1$) variant of Blatz and Ko's model [1], the function of specific potential energy takes the form

$$W = \frac{1}{2}\mu\Big(I_1 + 2\frac{1}{\sqrt{I_3}} - 5\Big),$$

and the boundary-value equilibrium problem is represented in the form

$$P'' = \frac{\psi^2 P^3 P'^3 r^2\gamma + 2PP'r^2 + P^3P'^3\gamma - 2P'^2r^3 - P'^4P^2r\gamma}{r^2P(2r + P'^3P\gamma)}, \quad P = \frac{r}{P'^3\gamma}, \quad r = r_0, r_1.$$

One can also construct an approximate (only terms that are quadratic relative to the displacement gradient in the governing relations are conserved) solution of the problem considered that allows one to discover the Poynting effect. Here the general expression for the relative elongation of the cylinder at small torsion angles has the form

$$\frac{\Delta l}{l} = \gamma - 1 \approx \frac{-2W_{,11}W_{,2} + W_{,3}W_{,11} + W_{,3}W_{,2} + W_{,12}W_{,2}}{W_{,2}(3W_{,11} + 2W_{,2})}\,\frac{J_p}{4S}\,\psi^2, \tag{8}$$

Figure 1 shows the dependences $\gamma(\psi r_1)$ for each model mentioned above. Calculations were performed for the ratio between the internal and external radii of the cylinder $r_0/r_1 = 0.01$. The solid and dashed curves refer to the numerical results and the calculation results obtained by formula (8), respectively. Figure 1a corresponds to the two-constant model for $\nu = 0.25$ (curves 1) and 0.49 (curves 2). Figure 1b corresponds to Blatz and Ko's three-constant model (curves 1 and 2 refer to the simplified and hypothetical variants, respectively). Figure 1c corresponds to Murnaghan's five-constant model (curves 1 and 2 refer to copper and 35KhGSA steel, respectively). The values of the Lamé and Murnaghan constants are taken from [1].

Figure 2 shows the dependences of the torsional moment on the magnitude of the torsion angle in the absence of elongation that were obtained numerically. Curve 1 corresponds to the hypothetical variant of Blatz and Ko's model, and curve 2 to Murnaghan's model for 35KhGSA steel. The values of the Lamé and Murnaghan constants for the latter are taken from [1]. The deviation of this dependence from the linear one is negligible for not too large ψr_1 for all the models considered except for the five-constant model. For example, for a 35KhGSA model of diameter 2 cm and length 10 cm, this deviation is approximately 10% if the but-ends are at an angle of 90° relative to each other.

The asymptotic relation (8) agrees with the results of numerical calculations and shows that the physically linear model allows one to make allowance for some specific features of the nonlinear theory (Poynting effect); however, in studying the deformation of elastic compressible media one should not restrict oneself only to its consideration. Indeed, for Murnaghan's model, expression (8) depends on four rather than two constants:

$$\frac{\Delta l}{l} = \gamma - 1 \approx -\frac{4\mu(2\mu + \lambda) + n\lambda + 4m\mu}{2\mu(3\lambda + 2\mu)}\,\frac{J_p}{4S}\,\psi^2.$$

This explains the qualitative difference in the behavior of the five-constant model (see Fig. 1c) and its two-constant approximation (see Fig. 1a).

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